On the Credibility of Prosecutorial (and other) Threats

Ehud Guttel and Shmuel Leshem

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Abstract

Principals use threats to make agents accept their demands. But what if agents outnumber threats? When negotiating with agents sequentially, a principal may have to forgo some agents to make threats against others credible. This paper examines a fundamental choice that such a principal faces: to divide agents and threats or to unite them. Using plea bargaining between limited-resource prosecution (principal) and offenders (agents), we show that by avoiding division the prosecution may boost—and never weaken—the credibility of its threats. We discuss the implications of this result for federalization of crime, employment negotiations, and formation of military alliances.

Keywords: Plea bargaining, Sequential negotiations, Credibility of threats

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1 Introduction

Prosecutors, employers, countries, and exasperated parents, to mention a few examples, use threats to motivate agents (offenders, employees, rival countries, and offsprings), but are often unable to follow through on their threats against all agents. Facing multiple agents but possessing only a few threats, such principals face a basic choice: to unite threats and agents or rather divide them.

To flesh out this choice, consider plea bargaining between prosecutors and offenders. Suppose $n$ offenders are arrested for committing a crime, but that the prosecution’s resources are sufficient for only $k$ trials ($k < n$). The prosecution, which seeks to maximize total punishment, can bring offenders to trial, sign plea agreements with them, or set them free. Clearly, the prosecution can try and convict $k$ offenders and release the other. To put additional offenders behind bars, the prosecution must make some offenders accept a plea deal.

Because the prosecution employs individual prosecutors to manage its caseload, it faces a choice on the optimal assignment of the $n$ cases. One alternative is to put in charge a single prosecutor with $k$ threats to negotiate plea deals with all $n$ offenders (uniting agents and threats). The other alternative is to distribute the $n$ cases among $k$ or fewer prosecutors, each with one or more threats, to negotiate plea agreements with offenders in their respective groups (dividing agents and threats). For example, the prosecution may assign $k$ prosecutors, each entrusted with resources to conduct one trial, to negotiate with $k$ distinct groups of offenders.

The prosecution’s unite-or-divide dilemma would not arise if prosecutors could simultaneously negotiate plea agreements with offenders. This is so because then a prosecutor with a single threat (ability to conduct one trial) could credibly threaten to bring to trial each offender and thereby make all $n$ offenders sign a plea deal. In particular, under simultaneous bargaining, the prosecutor should make each offender an (observable) plea offer, which is slightly less than the offender’s prospective sentence. The offender who expects the highest sentence accepts his plea offer, irrespective of other offenders’ decisions, for the prosecution will surely put him on trial if he declines the offer. But then the offender who expects the second-highest sentence must reason that he is next to stand trial should he turn down the prosecutor’s offer; this offender therefore accepts his plea offer too. Each remaining offender, in a descending order of expected sentences, similarly infers that he will be put on trial if he refuses the prosecutor’s offer and accordingly enters a plea agreement as well.

In practice, however, prosecutors are often unable to make simultaneous plea offers to offenders (for reasons we shortly explain), but rather must approach them one after the other. Under such circumstances, a single threat may not suffice to make all offenders settle. Consider, for example, a prosecutor who negotiates sequentially with three offenders ($n = 3$) expecting sentences of 6, 5, and 4. Suppose that, as offenders know, the prosecutor can bring only one offender to trial ($k = 1$). In this case, the prosecutor does not have a credible threat to put on trial the first offender, who expects a sentence of 6; this offender will consequently refuse a plea offer. This follows because if the offender expecting a sentence of 6 declines the plea offer, the prosecutor is better off releasing him and using her single threat to negotiate plea deals with the two subsequent offenders—
each of whom she can credibly threaten to take to trial. The best the prosecutor could do is releasing the lowest-sentence offender, and then negotiating with the other two offenders, who expect sentences of 6 and 5 (in this order). The maximum sum of sentences the prosecutor can impose is therefore 11. More generally, a prosecutor with one threat can negotiate plea deals with all offenders in a single group if and only if each offender’s sentence is greater than the sum of sentences of all the subsequent offenders.

The prosecutor’s optimal strategy under sequential negotiations may involve more nuanced trade-offs. Suppose that in addition to the three previous offenders, there is another offender who faces a sentence of 9.5. In this case, the prosecutor does not have a credible threat to put on trial either the first or the second offender in line. Knowing that the prosecutor would rather save her one threat to negotiate plea deals with subsequent offenders, both the first and the second offender would refuse to plead guilty. To maximize the sum of sentences obtained through plea bargains, the prosecutor should release the second offender (facing a sentence of 6). By doing so the prosecutor obtains a credible threat to put on trial each of the three other offenders—expecting sentences of 9.5, 5, and 4—each of whom will consequently sign a plea deal for a total of 18.5. The principle underlying the prosecutor’s optimal strategy under such circumstances is releasing the lowest-sentence offender (or offenders) such that the prosecutor has a credible threat to take to trial each remaining offender. Accordingly, in the presence of four offenders, the prosecutor’s best strategy might be releasing any one of the three lowest-sentence offenders or the two lowest-sentence ones.

It is the prosecution’s inability under sequential bargaining to settle with all offenders—and the resulting necessity to select among them—that gives rise to the divide-or-unite dilemma. If the prosecution can put on trial more than one offender ($k > 1$), yet cannot use a single threat to plea bargain with all $n$ offenders, it must decide whether to divide offenders and threats (by negotiating separately with different groups of offenders) or rather unite them (by negotiating with all offenders within a single group).

The intricate tradeoffs associated with the selection of offenders for plea bargaining seem to suggest that the resolution of the prosecution’s dilemma depends on the scope of its resources as well as on the number of offenders and their expected sentences. The paper’s key insight, however, is that uniting threats and offenders never weaken the prosecution’s bargaining position vis-à-vis offenders—irrespective of the number of threats and the characteristics of offenders (that is, for any $k$, $n$, and profile of sentences). Furthermore, while uniting threats and offenders never hurts the prosecution, it may embolden the prosecution’s bargaining position by boosting the credibility of its threats.

The logic that underlies these results is rooted in what makes threats credible. A prosecutor has a credible threat to put an offender on trial if the benefit of exercising the threat is greater than the corresponding cost. Thus, a threat is credible if the offender’s sentence is greater than the sum of sentences of offenders that the prosecutor must give

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1. Rather than bringing the first offender to trial, the prosecutor is better off releasing him along with the last offender, and extracting plea deals of 11 from the second and third offenders (expecting sentences of 6 and 5). For a similar reason, the prosecutor does not have a credible threat to put on trial the second offender.

2. Letting $x$ be the first offender’s sentence, the prosecution’s optimal strategy is to release the third and last offenders if $9 > x > 6$; the second offender if $10 > x > 9$; the third offender if $11 > x > 10$; and the last offender if $x > 11$. 

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up on when left with one less threat. We accordingly show that if $k$ threats are sufficient to make $n$ offenders in separate groups plead guilty, then uniting threats and offenders never increases the prosecution’s cost of exercising its last (i.e., $k$-th) threat against any of the $n$ offenders in a united group (Theorem 1). The benefit from uniting offenders and threats stems from the fact that the prosecution’s cost of exercising its $k$-th threat in a united group may be lower than the corresponding cost when negotiating with the same offender within his original group (Theorem 2).

To gain insight into the innocuousness of uniting threats and offenders, consider two prosecutors, each with one threat, who are able to make all offenders in their respective groups plead guilty. This implies that each offender’s sentence is greater than the sum of sentences of all the subsequent offenders in his group (i.e., offenders who face a lower sentence). Now suppose that one prosecutor with two threats negotiates single-handedly with all the offenders in a united group. Releasing all the offenders belonging to the same original group allows the prosecutor to use one threat to settle with the all the remaining offenders; for the remaining offenders belong to the other group and therefore would all yield to one threat. It follows that bringing to trial an offender in the united group never requires the prosecutor to give up on a sum of sentences greater than that of all the subsequent offenders in the same (original) group. The benefit of bringing to trial any offender in the united group (the offender’s sentence) is accordingly lower than the corresponding cost (the sum of sentences of the subsequent offenders in the same original group or less). The prosecutor thus has a credible threat to bring to trial every offender in the united group. The argument holds, by a similar reasoning, for any number of groups and threats.

Pooling together threats and offenders improves the prosecution’s bargaining power by lowering the opportunity cost of bringing offenders to trial. To illustrate this, consider again our previous example, and suppose that in addition to the three original offenders (6, 5, and 4) there are now three other offenders facing sentences of 16, 15, and 14. A prosecutor armed with two threats, negotiating with all six offenders in one group, can make all of them enter a plea agreement without exercising any of her threats. In particular, the prosecutor has a credible threat to bring to trial the first offender (16), because prosecuting this offender requires her to give up on the last three offenders, whose sum of sentences is only 15 ($6 + 5 + 4$). The prosecutor subsequently has a credible threat to bring to trial the second offender, because bringing this offender to trial requires her to give up on the last offender, whose sentence is 4. By the same logic, the prosecutor can credibly threaten the third offender because taking him to court again requires her to release the last offender.\(^3\) Having yet exercised any of her threats, each of the remaining three offenders in turn yields to the prosecutor as well.

By contrast, there is no division of the six offenders into two groups—each group negotiating with a prosecutor possessing one threat—such that all offenders accept a plea offer. Observe that a prosecutor with a single threat negotiating with three offenders can make all offenders plead guilty only if the first offender belongs to the high-sentence

\(^3\)After prosecuting the second offender, the prosecutor is left with offenders expecting sentences of 14, 6, 5, and 4; after prosecuting the third offender, she is left with offenders expecting sentences of 6, 5, and 4. In both cases, the best the prosecutor can do with one threat is releasing the last offender and settling with the remaining offenders.
triplet (16, 15, 14) and the other two belong to the low-sentence triplet (6, 5, 4). But then a prosecutor negotiating with the remaining three offenders would face two offenders belonging to the high-sentence triplet and one belonging to the low-sentence triplet. This prosecutor would therefore not be able to make all the remaining three offenders sign a plea agreement. Under any division of the six offenders, the prosecution must release one or two offenders to maximize total penalty.

We framed our inquiry in terms of the optimal matching between local prosecutors and offenders, where prosecutors do not transfer threats among themselves. On a larger scale, our investigation captures a basic choice on the institutional design of law enforcement. Two parallel systems try and penalize offenders: federal and state. An ongoing debate among policy makers and academics revolves around the allocation of authority between these systems. The debate is centered on Congress’ increasing tendency to enact federal criminal prohibitions, which are subject to the purview of federal enforcement agencies.

The resolution of the divide-or-unite dilemma illuminates an overlooked dimension of the choice between federal- versus state-based enforcement, which concerns the ability to induce offenders to admit guilt. A hallmark of the federal enforcement system is the centralization of prosecutorial resources and the clustering of offenders in a single pool. A state-based system, by contrast, distributes prosecutorial resources among states and concomitantly scatters offenders in separate pools. The choice between federal versus state enforcement thus embodies a choice between a centralized versus de-centralized prosecutorial system. This paper suggests that a federal system which unites threats and offenders has a strategic advantage over a state-based one: Whereas the former leverages the credibility of threats of putting offenders to trial, the latter erodes the credibility of such threats.

The scope of our inquiry extends beyond the confines of prosecutorial threats. The underlying contours of this paper’s analysis are sequential bargaining between one principal and multiple agents, in which the principal possesses too few threats to credibly threaten each and every one of the agents. As we later show, these features underlie negotiations between employers and workers with employment protection. They also characterize peace negotiations between rival countries, where each country’s military resources allow for only a single (or just a few) military engagement. The strategic benefit of uniting agents and threats suggests that firms with sufficiently large number of employees could exploit a right to lay off some of their workforce to induce workers to concede to wage reductions. In the international sphere, it sheds light on the motivation for forming military alliances and their circumstances.

Our paper is related to the literature on plea bargains as well as to the one on the credibility of threats. The prevalence of plea bargains has long been a puzzle. Knowing that prosecutors can only litigate a small number of cases, self-interested offenders should decline offers to plead guilty. The origin of the economic rationale for plea agreements goes back to the Supreme Court which underscored the cost saving associated with plea deals and the resulting advantage they provide for a resource-constrained enforcement agencies (e.g., Santobello v. New York). Extending this intuition, Landes (1971) highlighted

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[^4]: It suffices to consider an equal division (where each group consists of three offenders) for clearly no group of four (or more) offenders would all agree to a plea offer when the prosecutor is armed with only one threat.

the negative externalities that offenders impose on each other in entering plea deals. Subsequent literature explicated the significance of strategic rivalry among offenders for understanding the dynamics of plea negotiations with multiple offenders. Most notably, Easterbrook (1983) and Friedman (2001; pp. 91-2) argued informally that competition among offenders is key to facilitating plea bargains. More recently, Bar-Gill & Ben-Shahar (2009) suggested that the equilibrium of simultaneous plea negotiations is obtained by successive elimination of dominated strategies. The difficulty in this reasoning is that it implies, as we explained above, that irrespective of the magnitude of prosecutorial resources, the prosecution can make any number of offenders sign plea deals. Introducing sequential rather than simultaneous plea negotiations mitigates this difficulty and brings to the forefront the significance of the prosecution’s choice of whether to unite resources and offenders or rather divide them.\(^6\)

The notion that under simultaneous bargaining a single threat can be used to extract concessions from multiple agents has been suggested in other contexts. Dixit and Nalebuff (1991; p. 18) and Gintis (2009; p. 62) discuss a homeowner entitled under rent-control laws to evict one tenant in a rent-controlled building.\(^7\) Such homeowner, they show, can negotiate higher rents from all tenants by sending them simultaneous eviction notices coupled with an option to pay a higher rent. In a similar vein, Dari-Mattiacci and de Geest (2010) harnessed the fact that a single threat under simultaneous negotiations can be recycled to explain the prevalence of sticks (rather than carrots) as a means for inducing compliance by multiple agents. Although Dixit and Nalebuff pitched the rent-control hypothetical as involving sequential negotiations, neither they nor Gintis or Dari-Mattiacci and de Geest have addressed the potential difference between sequential versus simultaneous bargaining protocol.

More generally, this paper contributes to the extensive literature on strategies for making threats credible. Originating with Schelling (1956), this literature has studied mechanisms for increasing the cost (or lowering the benefit) of not exercising threats; or lowering the cost (or increasing the benefit) of exercising them.\(^8\) Classic examples include the folk wisdom of burning bridges behind an army to subdue an adversary and over-investing in capacity to deter entry (Dixit, 1980). Other well-known mechanisms are maintaining ignorance of the benefit from exercising a threat (Nalebuff, 1987) and sinking a portion of the cost of exercising it (Bebchuk, 1996; Fearon, 1997). The previous literature, however, has focused solely on strategies for bolstering the credibility of one threat. This paper, by contrast, deals with the efficient deployment of multiple threats.

The paper proceeds as follows. Section 2 derives foundational results for a sequence of \(n\) offenders to be credibly threatenable with \(k\) threats. Building on this section, Section 3 presents the main theorems of the paper, which together show that uniting threats and offenders never hurts the prosecution, but may well benefit it. Section 4 complements the analysis by presenting a necessary condition for a prosecution with \(k\) threats to be worse off upon dividing a credibly-threatenable sequence of \(n\) offenders into \(k\) subsequences and

\(^6\)Other papers that have examined the process and outcome of plea bargains are Grossman & Katz (1983), Reinganum (1988), and Backer and Mezzettie (2001). These papers, which study the effects of possible information asymmetries between prosecutors and offenders on plea negotiations, fall outside the scope of our analysis.

\(^7\)In New York and California a landlord of a rent-controlled building is entitled to evict at no cost one tenant (but only one) if he wishes to live in the building.

\(^8\)For a review, see Dixit and Nalebuff, 1991.
by finding a least upper bound on the prosecution’s percentage loss from such division (Theorem 3). We discuss three applications in Section 5.

2 A Model of a Few Threats and Multiple Agents

2.1 Setup

We consider a sequential game between a prosecutor and \( n \) offenders, where \( n \geq 3 \). The prosecutor seeks to meet out the maximum sum of sentences, but can only take \( k \) offenders to trial. We assume that \( k < n \) so that the prosecutor cannot bring to trial all the offenders. If offender \( i \) is taken to trial, he faces a sentence of \( x_i \). Let \( X = (x_1, ..., x_n) \) be a sequence of offenders, each represented by his sentence, listed in a decreasing order \( (x_i \geq x_{i+1}) \). We denote by \( X_i = (x_1, ..., x_n) \subseteq X \) the subsequence obtained by deleting the first \( i-1 \) terms of \( X \). Given that \( X \) is a decreasing sequence, \( X_i \) consists of offenders whose sentence is equal to or lower than \( x_i \). For example, if \( X = (6, 5, 4) \), then \( X_1 = (6, 5, 4) \), \( X_2 = (5, 4) \), and \( X_3 = (4) \).

The game proceeds in \( n \) stages. In stage 1, the prosecutor makes offender 1 a take-it-or-leave-it plea offer. If the offender accepts the offer, he pleads guilty and serves the agreed-upon sentence; the game then proceeds to the next stage. If the offender rejects the offer, the prosecutor can put the offender on trial, where he will be sentenced for \( x_1 \), or set the offender free. In each subsequent stage, the prosecutor makes offender \( i \) a plea offer, which the offender must either accept—and serve the corresponding sentence—or reject. If the offender rejects the offer and the prosecutor took \( k-1 \) or fewer offenders to trial in previous stages, the prosecutor can bring the offender to trial, where the offender will be sentenced for \( x_i \). The offender is otherwise set free. The game ends in stage \( n \).

Before delving into the analysis, we pause to explain the sequential nature of plea bargain negotiations. For one thing, a budget-constrained prosecution may not be able to simultaneously negotiate with multiple offenders. In the presence of limited resources, sequential negotiations are many prosecutors’ only alternative. For another, fundamental evidentiary rules discourage simultaneous bargaining. Having offenders testify against other offenders is often essential to establish guilt. However, an accomplice whose case has not been resolved yet (either by drop of charges or by conviction) can refuse to testify by invoking his Fifth Amendment privilege against self-incrimination. Thus under simultaneous negotiations, in which offenders can avoid testifying against one another, the prosecution may fail to have sufficient evidence to corroborate its allegations. Moreover, even if offenders were willing to waive their Fifth-Amendment privilege and testify against others prior to the resolution of their case, their testimony may bear little weight given the high risk of false testimony.\(^9\) By negotiating sequentially, prosecutors avoid these evidentiary hurdles and can thereby obtain vital evidence.

\(^9\)As the Supreme Court has noted, a defendant who is awaiting trial has “a special interest in lying in favor of the prosecution,” hoping that the prosecution would reward his collaboration in his own case. In addition, if the person testifying is an accomplice, he may minimize his own responsibility by inflating that of his partners. Such testimonies are accordingly treated with suspicion. Washington v. Texas, 388 U.S. 14, 87 (1967).
2.2 Threatenable Sequences

We say that a sequence of offenders is credibly threatenable (or simply threatenable) if the prosecutor can make every offender accept a plea offer equal to his respective sentence. A sequence of offenders is threatenable therefore if the prosecutor’s strategy of offering \( x_i \) in stage \( i = 1, \ldots, n \) and offender \( i \)’s strategy of accepting the prosecutor’s offer constitute a subgame perfect Nash Equilibrium. We shall assume that offenders would accept an offer equal to their sentence if they are indifferent between accepting and rejecting it. We shall further assume that the prosecutor would rather prosecute an offender if she is indifferent between prosecuting that offender and not prosecuting him.

We begin by presenting a necessary and sufficient condition for a sequence to be threatenable when \( k = 1 \); namely, when the prosecutor can only bring one offender to trial:

A sequence \( X \) is threatenable with one threat iff \( x_i \geq S(X_{i+1}) \) for all \( i < n \), \hspace{1cm} (1)

where \( S(\cdot) \) is the sum of the terms in \( \cdot \).

The condition in (1) ensures that if the prosecutor can plea bargain with offenders \( i + 1, \ldots, n \) for their respective sentences, she has a credible threat to take to trial offender \( i \), the preceding offender. This follows because if offender \( i \) rejects a plea offer of \( x_i \), the prosecutor would rather prosecute him (for \( x_i \)) than plea bargain with all the subsequent offenders (for \( S(X_{i+1}) \)). Because the prosecutor clearly has a credible threat to take the last offender (offender \( n \)) to trial, the condition in (1) implies—by backward induction—that she has a credible threat to take each previous offender to trial as well.

We call the condition for a sequence to be threatenable with one threat "Condition \( \tau^1 \)" (‘\( \tau \)' for threatenability; ‘1’ for the number of threats), and a sequence that satisfies condition \( \tau^1 \) a "\( \tau^1 \) sequence."

**Example 1** The decreasing geometric sequence \( (2^{n-1}, \ldots, 2^n, \ldots, 1) \) is threatenable with one threat for any \( n \).

In a decreasing geometric sequence, \( S(X_{i+1}) = 2^{n-i} - 1 \) and therefore \( x_i > S(X_{i+1}) \) for every \( i < n \). Each term is thus greater than the sum of all the subsequent terms, thereby satisfying Condition \( \tau^1 \). The same applies to any decreasing geometric sequence \( \{a^{n-i}\}_{i=1}^n \) for \( a > 2 \) (i.e., a decreasing geometric series with a common ratio less than one half).\(^{10}\)

Now, a sequence that is not threatenable with one threat—and hence does not satisfy Condition \( \tau^1 \)—can be made threatenable with one threat by omitting some of its terms. In particular, let \( T^1(X) \) be the set of all subsequences of \( X \) that satisfy Condition \( \tau^1 \). Note that \( T^1(X) \) is not empty because any two-term subsequence of \( X \) satisfies Condition \( \tau^1 \).

Let \( X^{T(1)} \) be the element of \( T^1(X) \)—thus threatenable with one threat—which has the largest sum of terms.\(^{11}\) Let \( X^{D(1)} \) be the complement subsequence; that is, \( X^{D(1)} \) is the

\(^{10}\)Note that for \( a > 2 \), \( x_i = a^{n-i} > a^{n-i} - 1 > a^{n-i-1} = S(X_{i+1}) \).

\(^{11}\)Formally, \( X^{T(1)} := \{X^{T(1)} \in T^1(X) \mid S(X^{T(1)}) \geq S(Y'), \ Y' \in T^1(X)\} \). In case of a tie, choose \( X^{T(1)} \) according to the lexicographic order.
subsequence containing all the terms of X that are not included in $X^{T(1)}$. Upon excising $X^{D(1)}$ from X, therefore, the prosecutor is left with $X^{T(1)}$. Note that a sequence X satisfies Condition $\tau^1$ iff $X_i^{T(1)} = X_i$ for every i and therefore $X_i^{D(1)}$ is empty for every i.

The following example demonstrates the definitions of $X^{T(1)}$ and $X^{D(1)}$:

**Example 2** Suppose $X = (14, 6, 5, 4)$. Then:

$X_1 = (14, 6, 5, 4)$, $X_1^{T(1)} = (14, 6, 5)$, and $X_1^{D(1)} = (4)$;

$X_2 = (6, 5, 4)$, $X_2^{T(1)} = (6, 5)$, and $X_2^{D(1)} = (4)$;

$X_3 = (5, 4)$, $X_3^{T(1)} = (5, 4)$, and $X_3^{D(1)} = ()$;

$X_4 = (4)$, $X_4^{T(1)} = (4)$, and $X_4^{D(1)} = ()$.

We now turn to the case of $k = 2$; a prosecutor who can take two offenders to trial (and thus has two threats):

A sequence X is threatenable with two threats iff $x_i \geq S(X_{i+1}^{D(1)})$ for all $i < n$.  \(2\)

The condition in (2), similar to Condition $\tau^1$, ensures that a prosecutor who has a credible threat to take offender $i + 1, \ldots, n$ to trial, has a credible threat to take offender $i$ to trial as well. In particular, by taking offender $i$ to trial, the prosecutor gives up on $S(X_{i+1}^{D(1)})$; namely, the minimum sum of sentences that must be forgone to render the subsequence $X_{i+1}$ threatenable with one threat. If $x_i$ is greater than or equal to $S(X_{i+1}^{D(1)})$, it is worthwhile for the prosecutor to take offender $i$ to trial even at the cost of being left with only one threat.

We will call the condition in (2) "Condition $\tau^2$" (the superscript stands for the number of threats, which is now 2) and any sequence that satisfies this condition a "$\tau^2$ sequence." Any 3-term sequence ($n = 3$) trivially satisfies Condition $\tau^2$ and is therefore a $\tau^2$ sequence. Any 4-term sequence ($n = 4$) satisfies Condition $\tau^2$ as well.\(^{12}\) A 5-term sequence ($n = 5$) satisfies Condition $\tau^2$ if and only if $x_1 \geq x_4 + x_5.\(^{13}\)

The next example presents a sequence threatenable with two threats, but not with one.

**Example 3** The decreasing Fibonacci sequence ($\ldots, 8, 5, 3, 2, 1, 1$) is threatenable with two threats, but not with one, for any n.

In the decreasing Fibonacci sequence, every term—except the last two—is equal to the sum of the two subsequent terms. Therefore, every term—other than the last three—is lower than the sum of all the subsequent terms, in violation of Condition $\tau^1$. The

\(^{12}\)Because in any 4-term sequence, $X_2^{T(1)} \supseteq (x_2, x_3)$ and therefore $x_1 \geq x_4 \geq S(X_2^{D(1)})$.

\(^{13}\)If $x_1 < x_4 + x_5$ then $X_2^{T(1)} = (x_2, x_3)$. It follows that $X_2^{D(1)} = (x_4, x_5)$ and therefore that $x_1 < S(X_2^{D(1)})$. Conversely, if $x_1 \geq x_4 + x_5$, then $x_1 \geq S(X_2^{D(1)})$, because $x_4 + x_5 \geq S(X_2^{D(1)})$ for any 5-term sequence.
decreasing Fibonacci sequences does satisfy Condition $\tau^2$, however. This follows because $x_i = S(X_{i+2}) + 1$ for $i \leq n - 2$ (as a simple proof by induction shows). and $S(X_{i+3}) \geq S(X_{i+1})^{D(1)}$ (since $(x_{i+1}, x_{i+2})$ is threatenable with one threat). Because $S(X_{i+2}) > S(X_{i+3})$, it follows that $x_i = S(X_{i+2}) + 1 > S(X_{i+2}) > S(X_{i+1})^{D(1)}$.\(^{14}\)

We now define recursively the condition for a sequence to be threatenable with $k$ threats. Denote by "Condition $\tau^{k-1}$" the necessary and sufficient condition for a sequence to be threatenable with $k - 1$ threats, where Condition $\tau^1$ and Condition $\tau^2$ are defined in (1) and (2), respectively. Let $T^{k-1}(X)$ be the set of all subsequences of $X$ that satisfy Condition $\tau^{k-1}$. Let $X^{T(k-1)}$ be the element of $T^{k-1}(X)$ that has the largest sum of terms\(^{15}\) and $X^{D(k-1)}$ be the subsequence containing all the terms of $X$ other than those of $X^{T(k-1)}$. Note that $X^{D(0)} = X$, because to render a sequence $X$ threatenable with no threats, the prosecutor must drop all of the terms of $X$.

The following Lemma presents a necessary and sufficient Condition for a sequence to be threatenable with $k$ threats:

**Lemma 1** A sequence $X$ is threatenable with $k$ threats iff $x_i \geq S(X_{i+1}^{D(k-1)})$ for all $i < n$.

The condition in Lemma 1 ensures that a prosecutor who can take offenders $i + 1, \ldots, n$ to trial with $k$ threats, has a credible threat to take offender $i$ to trial as well. In particular, by taking offender $i$ to trial, the prosecutor uses up one threat and therefore can only plea bargain with offenders in $X_{i+1}^{T(k-1)}$. The difference between $S(X_{i+1})$ and $S(X_{i+1}^{T(k-1)})$, which is $S(X_{i+1}^{D(k-1)})$, is the minimum sum of sentences of offenders that the prosecutor must give up on when her number of threats decreases from $k$ to $k - 1$. If $x_i$ is greater than or equal to $S(X_{i+1}^{D(k-1)})$, the prosecutor would rather take offender $i$ to trial than spare her $k$-th threat to plea bargain with subsequent offenders. We will call a sequence threatenable with $k$ threats a "$\tau^k$ sequence."

The following corollary presents a sufficient condition for a sequence to be threatenable with $k$ threats.

**Corollary 1** Any sequence of $2k$ terms is threatenable with $k$ threats.

**Proof.** By induction. The corollary is clearly true for $k = 1$. Suppose the corollary is true for a sequence of $2j$ terms. Consider a sequence $X$ with $2j + 2$ terms. Then $X_3$ has $j$ terms and therefore, by the induction hypothesis, is threatenable with $j$ threats. In addition, because $X_3$ is a $\tau^j$ sequence, $X_3^{D(j)}$ is empty and therefore $x_2 \geq S(X_3^{D(j)})$. Furthermore, after omitting the last term of $X$ ($x_n$), the (modified) subsequence $X_2$ consists of $2j$ terms and is threatenable with $j$ threats. It follows that $x_n \geq S(X_2^{D(j)})$ and therefore that $x_1 \geq S(X_2^{D(j)})$ (because $x_1 \geq x_n$). This proves the induction step and completes the proof. □

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\(^{14}\)In the Fibonacci sequence example, the sum of any two consecutive terms is greater than the sum of all the subsequent terms (that is, $x_i + x_{i+1} \geq S(X_{i+2})$ for $i \leq n - 3$). As we later show, this is a necessary condition for a sequence to be threatenable with 2 threats.

\(^{15}\)Here too let the lexicographic order break ties.
The next corollary of Lemma 1 presents a necessary condition for a sequence to be threatenable with \( k \) threats:

**Corollary 2** If a sequence is threatenable with \( k \) threats, then the sum of its first \( k \) terms is weakly greater than the sum of all the subsequent terms.

**Proof.** By induction. The corollary is clearly true for \( k = 1 \): a sequence is threatenable with one threat iff every term is greater than or equal to the sum of all the subsequent terms. Suppose, for the induction hypothesis, that the corollary is true for a \( \tau^j \) sequence. Consider a \( \tau^{j+1} \) sequence \( X \), which satisfies Condition \( \tau^{j+1} \); in particular, \( x_1 \geq S(X_2^{D(j)}) \) (*). Now, a simple proof by induction (which we omit) shows that a prosecutor with \( k \) threats always keeps the first \( k \) offenders. This fact together with the induction hypothesis imply that \( x_2 + \ldots + x_{j+1} \geq S(X_2^{T(j)}) - (x_2 + \ldots + x_{j+1}) \) (**). The left-hand side consists of the first \( j \) terms of \( X_2^{T(j)} \); the right-hand side includes the remaining terms of \( X_2^{T(j)} \). Adding up the inequalities (*) and (**) yields \( x_1 + \ldots + x_{j+1} \geq S(X_2^{D(j)}) + S(X_2^{T(j)}) - (x_2 + \ldots + x_{j+1}) \) (**). But \( S(X_2^{D(j)}) + S(X_2^{T(j)}) = S(X_2) \) and therefore (***) reduces to \( x_1 + \ldots + x_{j+1} \geq S(X_{j+2}) \). This proves the induction step and the corollary. \( \blacksquare \)

Before concluding this section, we consider minimal threatenable sequences. A minimal \( n \)-term \( \tau^k \) sequence is, roughly, the lowest-terms sequence threatenable with \( k \) threats whose last term is 1. For example, a minimal \( \tau^1 \) sequence is \((2^{n-2}, \ldots, 8, 4, 2, 1, 1)\), where each term is equal to the sum of the subsequent terms. More precisely, a \( \tau^k \) sequence \( X \) is minimal iff \( X \) is threatenable with \( k \) threats, \( x_n = 1 \), and \( S(X_i) \leq S(Y_i) \) for all \( i < n \) and any \( \tau^k \) sequence \( Y \) with \( y_n = 1 \). An equivalent formal definition follows:

**Definition 1** A minimal \( \tau^k \) sequence is a sequence whose last \( 2k \) terms are 1 and each preceding term is \( x_i = S(X_{i+1}^{D(k-1)}) = 2x_{i+k} \).

The last \( 2k \) terms of a minimal \( \tau^k \) sequence are 1, because, by Corollary 1, any \( 2k \)-term sequence is threatenable with \( k \) threats. Each preceding term is equal to the minimum sum of terms that must be dropped to render the subsequence beginning with the next term (all through the last term) threatenable with \( k-1 \) threats (i.e., \( S(X_{i+1}^{D(k-1)}) \)). Because \( S(X_{i+1}^{D(k-1)}) \) decreases with \( i \), a minimal threatenable sequence is (weakly) decreasing. Moreover, as we show in the Appendix, in a \( \tau^k \) minimal threatenable sequence, \( x_i = 2x_{i+k} \) for \( x_i \neq 1 \). For example, in a minimal \( \tau^2 \) sequence each non-1 term is equal to twice the term that follows the subsequent term: \((\ldots, 4, 4, 2, 1, 1, 1, 1, 1)\). It follows straightforwardly that the sum of the first \( k \) terms of any minimal threatenable sequence of \( 2k \) or more terms is equal to the sum of all the subsequent terms. Any such sequence therefore satisfies the consequent in Corollary 2 as an equality.

3 Uniting and Dividing Threatenable Sequences

3.1 Uniting Threatenable Sequences

In this section we consider the consequences of uniting and dividing threatenable sequences. We denote the union of two sequences, \( X \) and \( Y \), as the monotone decreasing
sequence of all the terms of $X$ and $Y$. We begin by showing that uniting threatenable sequences and their corresponding threats never hurts the prosecution.

**Theorem 1** Let $X$ be a sequence threatenable with $q$ threats and $Y$ be a sequence threatenable with $r$ threats. Then the sequence $X \cup Y$ is threatenable with $q + r$ threats.

**Proof.** We proceed in three steps. In step 1, we show that the union of two sequences, each threatenable with one threat, is threatenable with two threats. In step 2, we show that the union of two sequences, one threatenable with one threat and the other with any number of threats, is threatenable with the sum of the threats of the constituent sequences. The proof concludes in step 3, which proves the Theorem.

**Step 1:** Suppose $X$ and $Y$ are each threatenable with one threat and let $Z = X \cup Y$. Suppose that $z_i = x_j \in X$, where $i \leq n - 4$ (by Corollary 1, the last 4 terms of $Z$ are threatenable with two threats). Let $Z'_{i+1}$ denote the subsequence obtained by deleting from $Z_{i+1}$ all the terms of $X$ $(x_{j+1}, x_{j+2}, \ldots, x_n)$, whose sum is $S(X_{j+1})$. The subsequence $Z'_{i+1}$ thus consists only of terms that belong to $Y$. Because $Y$ is threatenable with one threat, so is $Z'_{i+1}$. But $Z'_{i+1} \subseteq Z^{(1)}_{i+1}$ (by definition of $Z^{(1)}_{i+1}$) and therefore $S(X_{j+1}) \geq S(Z^{(1)}_{i+1})$. Moreover, $x_j \geq S(X_{j+1})$ because $X$ is threatenable with one threat and therefore $x_j \geq S(Z^{(1)}_{i+1})$. Because $z_i = x_j$, it follows that $z_i \geq S(Z^{(1)}_{i+1})$. The same argument holds if $z_j \in Y$. Thus $Z$ satisfies Condition $\tau^2$ and is accordingly threatenable with two threats.

**Step 2:** By the proof of step 1, the Theorem is true for $q = r = 1$. Suppose—for the induction hypothesis—that the Theorem is true for $q = 1$ and $q = 1, 2, \ldots, \hat{r}$. Then the Theorem is true for $q = 1$ and $r = 1, 2, \ldots, \hat{r} + 1$ as well. To prove the induction step, let $X$ be a sequence threatenable with one threat and $Y$ be a sequence threatenable with $\hat{r} + 1$ threats. If $z_i = x_j \in X$, the proof proceeds along the lines of Step 1. Suppose then that $z_i = y_j = Y$. Let $Z'_{i+1}$ denote the subsequence obtained by deleting from $Z_{i+1}$ all the terms in $Y_j^{D(\hat{r})}$. The subsequence $Z'_{i+1}$ thus consists of $Y_j^{T(\hat{r})}$ (which is threatenable with $\hat{r}$ threats) and terms that belong to $X$ (which is threatenable with one threat). By the induction hypothesis, $Z'_{i+1}$ is threatenable with $\hat{r} + 1$ threats, whence $S(Y_j^{D(\hat{r})}) \geq S(Z^{(1)}_{i+1})$. But $Y$ is threatenable with $\hat{r} + 1$ threats and therefore $y_j \geq S(Y_j^{D(\hat{r})})$. We consequently obtain $z_i = y_j \geq S(Y_j^{D(\hat{r})}) \geq S(Z_{i+1}^{D(\hat{r}+1)})$. It follows that $Z$ satisfies Condition $\tau^{\hat{r}+2}$ and is accordingly threatenable with $\hat{r} + 2$ threats.

**Step 3:** By Step 2, the Theorem is true for $q = 1$ and any $r$. We now show that if the Theorem is true for $q = 1, 2, \ldots, \hat{q}$ and any $r$ (note that Step 2 proves this for $\hat{q} = 1$), then it is true for $q = 1, 2, \ldots, \hat{q} + 1$ and any $r$ as well. To prove the induction step, let $X$ be a sequence threatenable with $\hat{q} + 1$ threats and $Y$ be a sequence threatenable with $r$ threats. If $r \leq \hat{q}$, then, by the induction hypothesis, the union of $X$ and $Y$ is threatenable with $\hat{q} + r + 1$ threats. Suppose then that $r > \hat{q}$. If the term of the union belongs to $X$, the proof follows Step 2 straightforwardly. If the term of the union belongs to $Y$, the proof follows by an additional induction argument. Under this induction argument, if the Theorem is true for $q = 1, 2, \ldots, \hat{q} + 1$ and $r = 1, \ldots, \hat{r}$, where $\hat{r} \geq \hat{q}$ (by the previous induction argument this holds for any $q$ and $r = 1, \ldots, \hat{q}$), then it is true for $q = 1, 2, \ldots, \hat{q} + 1$ and $r = 1, \ldots, \hat{r} + 1$. The proof of this induction argument follows Step 2 as well. ■
Consider the case of two prosecutors, each with one credible threat, who can make all the offenders in their respective groups, X and Y, enter a plea agreement. Recall that a prosecutor with one threat can credibly threaten an offender to put him on trial iff that offender’s sentence is greater than the sum of sentences of all the subsequent offenders (who expect lower sentences); for prosecuting any offender leaves the prosecutor with no threat.

Suppose that one prosecutor with two threats negotiates with offenders in Z, the union of X and Y. The prosecutor’s cost of putting on trial an offender in X—and thereby having only one threat left—is no greater than the sum of sentences of all the subsequent offenders in X. This follows because, by dropping charges against all the subsequent offenders in X, only offenders in Y—which is threatenable with one threat—are left. Because the sentence of any offender in X is greater than the sum of sentences of all the subsequent offenders, the prosecutor’s loss from bringing to trial any offender in X is lower than that offender’s sentence. The same argument holds if the offender belongs to Y.

A simple induction argument (in step 2) shows that the union of two sequences, one threatenable with one threat and the other with any number of threats, is threatenable with the sum of threats of the constitutive sequences. A slightly more involved induction argument (in step 3) shows that the union of two sequences is threatenable with the sum of threats of the constituent sequences. Theorem 1 can readily be generalized to the union of more than two sequences (we omit the proof).

3.2 Dividing Threatenable Sequences

Theorem 1 shows that uniting threatenable sequences never hurts the prosecutor. The next theorem shows, by contrast, that dividing a threatenable sequence may weaken the prosecution’s bargaining power.

**Theorem 2** For all \( k \geq 2 \) there exists a sequence \( U \) such that \( U \) is threatenable with \( k \) threats but cannot be divided into \( k \) subsequences such that every subsequence is threatenable with one threat.

**Proof.** By construction. Let \( T \) be a sequence threatenable with \( k \) threats, but not with \( k-1 \) threats. Let \( H \) be a sequence of \( k+1 \) terms such that (i) the first term of \( H \) is weakly greater than the sum of the terms of \( T \) (\( h_1 \geq S(T) \)); and (ii) the difference between any two terms of \( H \) is less than the last term of \( T \) (\( h_i \geq h_1 - t_n \) for \( i = 2, \ldots, k+1 \)). We show that the sequence \( U = H \cup T \) is threatenable with \( k \) threats, but cannot be divided into \( k \) subsequences such that every subsequence is threatenable with one threat.

We begin by showing that the last term of \( H \) is greater than the first term of \( T \) (\( h_{k+1} > t_1 \)). Recall that, by construction, \( h_1 \geq S(T) \) and \( h_{k+1} > h_1 - t_n \). These inequalities together imply that \( h_{k+1} > S(T) - t_n \). But \( S(T) > t_1 + t_n \) (because \( |T| \geq 3 \)) in turn implies that \( S(T) - t_n > t_1 \). It follows that \( h_{k+1} > S(T) - t_n > t_1 \). Moreover, because \( S(T_2) = S(T) - t_1 \) and \( t_1 \geq t_n \), we have \( h_{k+1} > S(T_2) \). Thus, every term of \( H \) is strictly greater than the sum of all the terms of \( T \) less \( t_1 \).
Next, we show that the union $U = H \cup T$ is threatenable with $k$ threats. Because $T$ is threatenable with $k$ threats, it remains to show that every $u_i \in H$ satisfies Condition $\tau^k$; that is, $u_i \geq S(U^{D(k-1)}_{i+1})$ for $i = 2, \ldots, k + 1$. To see this, observe that to render the subsequence $U_{i+1}$, where $i = 2, \ldots, k + 1$, threatenable with $k - 1$ threats, the prosecutor need not give up on more than $S(T_2)$, which in turn implies that $S(U^{D(k-1)}_{i+1}) \leq S(T_2)$. This follows because the subsequence that is left after dropping from $U_{i+1}$ all the terms of $T_2$ consists of $k$ (for $i = 2$) or fewer terms (for $i = 3, \ldots, k + 1$) and therefore, by Corollary 1, is threatenable with $k/2$ threats. But $k - 1 \geq k/2$ for $k \geq 2$ and therefore the subsequence left after dropping from $U_{i+1}$ all the terms of $T_2$ is threatenable with $k - 1$ threats as well. Because $u_i \geq S(T_2)$ for every $u_i \in H$ we have $u_i \geq S(U^{D(k-1)}_{i+1})$ for $i = 2, \ldots, k + 1$. Now, to render $U_2$ threatenable with $k - 1$ threats, the prosecutor must never drop more than $S(T)$; for the subsequence that remains after dropping from $U_2$ all the terms of $T$ consists of $k$ terms ($u_2, \ldots, u_{k+1}$) and is therefore threatenable with $k - 1$ threats. But $u_1 \geq S(T)$ by construction, whence $u_1 \geq S(U^{D(k-1)}_2)$.

Finally, we show that there is no division of $U$ into $k$ subsequences such that every subsequence is threatenable with one threat. Recall that the difference between any two terms of $H$ is smaller than $t_n$. It follows that no subsequence containing 2 terms of $H$, along with any other term of $U$, is threatenable with one threat. Because the prosecutor must divide $U$ into $k$ subsequences, by the Pigeonhole Principle at least one subsequence contains two terms of $H$. But then $T$ must be divided into (the remaining) $k - 1$ subsequences so that each subsequence is threatenable with one threat. By Theorem 1, however, $T$ cannot be so divided, for otherwise $T$ would be threatenable with $k - 1$ threat. Upon dividing $U$ into $k$ subsequences, therefore, the prosecutor must drop a total sum of terms of $S(T^{D(k-1)}_1)$.

The idea behind the proof of Theorem 2 is as follows. The sequence $U$ consists of a tail sequence ($T$) and a head sequence ($H$). The tail sequence, $T$, is threatenable with $k$, but not fewer, threats. The head sequence, $H$, has exactly $k + 1$ terms. In addition, the first term of $H$ is greater than the sum of all the terms of $T$, and the difference between any two terms of $H$ is strictly less than the last term of $T$. These properties of $H$ ensure that (i) the last term of $H$ ($h_{k+1}$) is strictly greater than the first term of $T$ ($t_1$), (ii) the union of $H$ and $T$ is threatenable with $k$ threats, and (iii) there is no division of $H$ into $k$ subsequences such that each subsequence is threatenable with one threat and all subsequences have exactly one term. Now, given than one subsequence of the division of $U$ into $k$ subsequences each threatenable with one threat includes two terms of $H$—and therefore cannot admit any additional term and still be threatenable with one threat—the remaining $k - 1$ subsequences must absorb all the terms of $T$. But $T$ is not threatenable with $k - 1$ threats, and therefore, by Theorem 1, cannot be divided into $k - 1$ subsequences such that every subsequence is threatenable with one threat. Partitioning $U$ into $k$ subsequences thus forces the prosecutor to give up on the minimum sum of terms required to render $T$ threatenable with $k - 1$ threats ($S(T^{D(k-1)})$).

To illustrate Theorem 2, consider the motivating example in the Introduction, where $U = (16, 15, 14, 6, 5, 4)$ and $k = 2$. The tail sequence in this example is $T = (6, 5, 4)$, which is threatenable with two threats, but not with one. The head sequence, $H = (16, 15, 14)$, has 3 terms ($k + 1$), its first term (16) is greater than the sum of the terms of $T$ (15), and the difference between any two terms of $H$ is less than the last term of $T$ (4). Upon dividing $U$ into 2 subsequences, the prosecutor must give up on the last term of $T$ (4);
by dropping this term the subsequence $T$ becomes threatenable with one $(k - 1)$ threat.

This example sheds light on the source of the prosecution's benefit from uniting threats and offenders. In the united sequence $U = (16, 15, 14, 6, 5, 4)$, the prosecution's cost of exercising its second threat against the first term of the subsequence $U_4 = (6, 5, 4)$ is 0, because $U_{5}^{D(1)}$ is empty. Now, under any division of $U$ into two subsequence $H$ and $T$, the entire tail sequence $T$ must follow a single term of $H$ (for the other subsequence consists of two terms of $H$). The cost of exercising a single threat against the first term of the tail subsequence $T = (6, 5, 4)$ is accordingly 9, because $T_{2}^{D(0)} = (5, 4)$. The benefit from uniting threats and offenders thus stems from lowering the cost of exercising the marginal threat in the united sequence relative to the constituent sequences.

We say that a $\tau^k$ sequence is partitionable if it can be divided into $k$ subsequences, each threatenable with one threat (the opposite holds for an unpartitionable sequence). A natural question concerns the prosecution's percentage loss from dividing an unpartitionable $\tau^k$ sequence into $k$ subsequences. In the motivating example in the Introduction, the percentage loss from division is $1/15$, because the sum of all offenders' sentences is 60 and division requires the prosecutor to release the last offender, whose sentence is 4. But the prosecutor could clearly lose more (percentage-wise). For example, suppose that $T$ is the 3-term sequence $(1, 1, 1)$ and that $H$ is the 3-term sequence $(3, 2 + \delta, 2 + \delta)$, where $\delta \in (0, 1]$. Note that both $H$ and $T$ satisfy the conditions in the proof of Theorem 2 and therefore the union $H \cup T$ is threatenable with two threats but cannot be divided into two $\tau^1$ subsequences. The prosecutor's percentage loss from division is $\frac{1}{10 + 2\delta}$, which is bounded above by $\frac{1}{10}$. This, in fact, holds for any $\tau^2$ sequence of 6 terms: The least upper bound of the prosecution's percentage loss from dividing a 6-term $\tau^2$ sequence into two $\tau^1$ subsequences is $\frac{1}{10}$. We take up this question more systematically in the next section.

4 The Loss from Dividing Threatenable Sequences

In this section we present a least upper bound on the prosecutor's percentage loss from dividing a $\tau^k$ sequence into $k$ subsequences. Our aim is twofold. First, we wish to gain insight into whether the maximum loss from division increases or decreases with the number of threats, $k$. Second, in the course of our analysis, we present a necessary condition for an arbitrary $\tau^k$ sequence to be unpartitionable as well as the maximum absolute loss from dividing such a sequence. We first consider the case of $k = 2$; we then generalize the results to an arbitrary $k$.

We begin by stating a sufficient condition for a $\tau^2$ sequence $X$ to be partitionable:

If $X$ is a $\tau^2$ sequence and $X_4$ a $\tau^1$ sequence, then $X$ is partitionable. \hfill (3)

That is, if $X$ is threatenable with two threats and $X_4$ is threatenable with one threat, then $X$ is partitionable. (1) thus implies that any 5-term $\tau^2$ sequence is partitionable, because in a 5-term sequence $X_4 = (x_4, x_5)$ is a $\tau^1$ sequence. The implication in (3) follows because if $X_4$ is threatenable with one threat, then $X$ can be divided into two $\tau^1$ subsequences, $X_2^{T(1)}$ and $x_1 \cup X_2^{D(1)}$ (note that $X_2 = X_2^{T(1)} \cup X_2^{D(1)}$). In particular, $X_2$
is a $\tau^1$ sequence by definition. As for $x_1 \cup X_2^{D(1)}$, observe that $x_1 \geq X_2^{D(1)}$ by Condition $\tau^2$. As we now show, if $X_4$ is threatenable with one threat, so is $X_2^{D(1)}$.

Suppose that $X_4$ is threatenable with one threat. If $x_3 \in X_2^{T(1)}$, then $X_2^{D(1)} \subseteq X_4$ (because $x_2 \in X_2^{T(1)}$). But $X_4$ is threatenable with one threat and therefore so is its subsequence $X_2^{D(1)}$. If, on the other hand, $x_3 \notin X_2^{T(1)}$, then $x_4 \in X_2^{T(1)}$ (because $X_4$ being threatenable with one threat implies that $x_4 \geq S(x_3)$). Because $x_2 \in X_2^{T(1)}$ and $x_3 \notin X_2^{T(1)}$, the first term of $X_2^{D(1)}$ is $x_3$. But the sum of all the terms of $X_2^{D(1)}$ less $x_3$ is (weakly) lower than $x_4$ (because $X_4$ is threatenable with one threat), and therefore this sum is lower than $x_3$ as well. Thus, in this case too, if $X_4$ is threatenable with one threat, so is $X_2^{D(1)}$.

It follows directly from (3) that

$$\text{The maximum absolute loss from dividing a } \tau^2 \text{ sequence } X \text{ is } S(X_4^{D(1)}). \quad (4)$$

By dropping all the terms of $X_4^{D(1)}$, the subsequence $X_4$ of the resulting sequence becomes a $\tau^1$ sequence and therefore satisfies the antecedent in (1). Upon dividing a $\tau^2$ sequence into two subsequences, therefore, the prosecutor never loses more than the minimum sum of terms that must be dropped to render $X_4$ threatenable with one threat.

We present an additional preliminary result. For any $\tau^2$ sequence $X$ of 3 or more terms we have

$$S(X^{T(1)}) \geq S(X^{D(1)}). \quad (5)$$

The inequality in (5) follows because, by Corollary 2, $x_i + x_{i+1} \geq S(x_{i+2})$ for any $\tau^2$ sequence. But $(x_i, x_{i+1})$ is itself a $\tau^1$ sequence and therefore $S(X_1^{T(1)}) \geq S(x_{i+2}) \geq S(X_i^{D(1)})$.

Observe that (5) is satisfied as equality iff $X$ is a minimal $\tau^2$ sequence and $n \geq 4$ and even. For example, $X = (2, 2, 1, 1, 1, 1)$ consists of $X^{T(1)} = (2, 2)$ and $X^{D(1)} = (1, 1, 1, 1)$, where $S(X^{T(1)}) = S(X^{D(1)}) = 4$. Similarly, $X = (4, 4, 2, 2, 1, 1, 1)$ consists of $X^{T(1)} = (4, 4)$ and $X^{D(1)} = (2, 2, 1, 1, 1, 1)$, where $S(X^{T(1)}) = S(X^{D(1)}) = 8$.

We now proceed to construct an unpartitionable $\tau^2$ sequence $U$ such that $S(U_4^{D(1)}) = S(U_4^{T(1)})$ and all the terms of $U_4^{D(1)}$ must be dropped upon dividing $U$ into two subsequences (otherwise the percentage loss from dividing would not be maximal). For such a sequence $U$ we have

$$S(U) > 4S(U_4) - 2u_n, \quad (6)$$

where $u_n$ is the last—and therefore smallest—term of $U$. To see why (6) holds, observe first that the prosecutor must give up on all the terms of $U_4^{D(1)}$ iff $u_1 - u_3 < u_n$. For if $u_1 - u_3 \geq u_n$, then $U$ could be divided into $(u_1, u_3, u_n)$ and $U_2 \cup U_4^{T(1)}$, each threatenable with one threat. The fact that $u_1 - u_3 < u_n$ in turn implies that $U_2^{T(1)} = (u_2, u_3)$ and therefore that $U_2^{D(1)} = U_4$. By Condition $\tau^2$, we have $u_1 \geq S(U_2^{D(1)}) = S(U_4)$. Because $u_2$ and $u_3$ must each be greater than $S(U_4) - u_n$, the sum of the 3-term head sequence

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16Because $u_2 \geq S(U_3^{D(1)})$, by Condition $\tau^2$, and $S(U_3^{D(1)}) \geq S(U_4^{D(1)}) = S(U_4^{T(1)})$, it follows that $u_2 \geq S(U_4^{T(1)})$. 

of $U$ is accordingly (strictly) greater than $S(U_4) + 2(S(U_4) - u_n)$, which simplifies to $3U_4 - 2u_n$. Because the sum of all the terms of the tail subsequence is $S(U_4)$, we have $S(U) > 4S(U_4) - 2u_n$.

Finally, $S(U_4) \geq \frac{1}{2}S(U_4^{D(1)})$ from (5) and hence $S(U) > 8S(U_4^{D(1)}) - 2u_n$. Because $2u_n << 4U_4$ for a sufficiently large $n$, it follows that

$$\text{The least upper bound of the percentage loss from dividing a } \tau^2 \text{ sequence is } \frac{1}{8}. \quad (7)$$

The next Theorem considers the least upper bound of the percentage loss from dividing an arbitrary threatenable sequence:

**Theorem 3** The least upper bound of the percentage loss from dividing a $\tau^k$ sequence into $k$ subsequences is decreasing with $k$ and is equal to $\frac{k-1}{2k^2}$.

The proof is relegated to the Appendix, but its outline follows the case of $k = 2$. We first show that a necessary condition for a $\tau^k$ sequence to be unpartitionable is that the tail subsequence $X_{2k}$ is not threatenable with one threat; this in turn extends the implication in (3) to a $\tau^k$ sequence. This immediately implies that the maximum absolute loss of dividing a $\tau^k$ sequence into $k$ subsequence is $S(X_{2k}^{D(1)})$, thereby generalizing (4). Finally, we show that the maximum percentage loss from rendering a $\tau^k$ sequence threatenable with one threat is $\frac{k-1}{k}$. This in turn generalizes (5) to a $\tau^k$ sequence (note that $\frac{k-1}{k} = \frac{1}{2}$ for $k = 2$).

We proceed by constructing a $\tau^k$ tail sequence, $U_{2k}$, which involves the maximum percentage loss upon rendering it threatenable with one threat; this subsequence is a minimal threatenable sequence with $k$ threats. We then construct a head subsequence of $2k - 1$ terms, whose first term, $u_1$, is equal to the sum of the terms of the tail subsequence, $S(U_{2k})$. All the other terms of the head subsequence are each equal to $(u_1 - u_n) + \varepsilon$ (recall that $u_1$ is the first term of the head subsequence and $u_n$ the last term of the tail subsequence). The least upper bound of the percentage loss from division is obtained by setting $\varepsilon = 0$ so that the difference $u_1 - u_i$ equals $u_n$ for $i = 2, \ldots, 2k - 1$.

Upon dividing $U$ into $k$ subsequence the prosecutor loses $S(U_{2k}^{D(1)}) = \frac{k-1}{k}S(U_{2k})$; the sum of terms that must be dropped to render the minimal $\tau^k$ sequence $U_{2k}$ threatenable with one threat. As the number of terms of $U_{2k}$ goes to infinity, each of the $2k - 1$ terms of the head subsequence approaches $S(U_{2k})$ and the sum of all the terms of $U$ approaches $2kS(U_{2k})$. The percentage loss from division is accordingly bounded above by $\frac{(k-1)/k}{2k}$.

### 5 Applications

In this section we present three applications in which principals benefit from uniting agents and threats. One implication, as briefly noted, concerns the institutional design of the criminal system. We additionally discuss employment relations in times of economic downturn and peace negotiations between rival countries.
From 1970 to 1995, federal criminal provisions have increased by nearly 40%, and the number of criminal cases handled by the federal system has roughly doubled. According to recent data, federal criminal legislation has continued to grow as Congress enacts an average of 56.5 new crimes a year (Baker, 2008). This expansion in federal regulation of criminal activities has been widely criticized, and a professional task force has concluded that “the Congressional appetite for new crimes ... is not only misguided and ineffectual but has serious adverse consequences ...” (American Bar Association, 1998).

A particularly noteworthy critique of federal involvement has pointed out the advantage of a decentralized system in allocating public goods such as law enforcement. According to this claim, which is grounded in Tiebout’ s (1959) influential work, a decentralized system makes localities compete in attracting capital and consumers. This competition stimulates states to avoid inefficiencies (fighting crime optimally) as well as to align expenditures to taxpayers’ preferences (choosing the proper investment of public resources in crime prevention), thereby increasing social welfare.17

Our analysis suggests, however, that a decentralized system comes with a cost. In the presence of scarce prosecutorial resources, prosecutors must rely on plea bargains to have offenders punished. A centralized system, in which prosecutorial resources and offenders are pooled together, maximizes prosecutors’ bargaining power. A decentralized system can be particularly inefficient if the distribution of offenders in different states is random. In section 4, we have shown the advantage of a centralized system versus a decentralized one in the case in which offenders are placed in optimal groups, so that the loss from division is minimal. If offenders were grouped otherwise, however, the actual loss from a decentralized system would be greater. The choice between a state-based versus a federal-based system accordingly requires balancing the benefits of Tiebout’s-like competition against those of enhancing prosecutors’ negotiation leverage.

Labor markets provide another example in which principals employ threats in negotiations with agents. As with plea bargaining, here too principals are limited in their ability to use threats. This limitation takes the form of regulation intended to protect agents. The right to discharge employees, for example, is now commonly subject to a "just cause" restriction, which requires employers to provide justification for terminating workers’ contracts (it is estimated that around one-third of all employees in the US enjoy such protection, Verkerke 2014; pp. 54-5). One important justification is "adverse business conditions," which usually reflect a decline in the demand for the employer’s services or products. This rule has been perceived as balancing between workers and employers’ legitimate needs, providing stability for workers while allowing employers to avoid inefficiency as markets conditions deteriorate (Restatement 3rd of Employment Law § 2.04, reporters notes cmt. e).

This standard view of the rule disregards, however, the possibility of strategic behavior by employers. Layoff threats allow employers to impose salary cuts on protected employees (see, for example, Myers 1994). Although the rule formally permits discharging only "unnecessary" workers, while firmly protecting other workers’ contractual rights, an employer might invoke it to exploit workers in bad economic times. If an employer can

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17Interestingly, almost three decades before Tiebout, the Supreme Court in New State Ice Co. v. Liebman alluded to the benefit of letting states compete and design local policies: “It is one of the happy incidents of the federal system that a single courageous state may, if its citizens choose . . . try novel social and economic experiments . . .” 285 U.S. 262, 311 (1932).
negotiates with her employees simultaneously, a right to discharge one worker may suffice to induce all workers to accept a salary reduction. Starting with the top-paid worker, each worker in turn reasons that he would lose his job should he refuse to renegotiate his promised salary. The employer consequently can reduce workers’ salaries to their opportunity wage.

This dynamic may unravel if employers can only approach workers one after the next (which seems more plausible as the number of workers rises). As with plea negotiations, employers may benefit from not executing their threats against higher-ranked employees so that they can use them against subsequent, lower-ranked ones. To maximize their profits, employers may have to discriminate among employees, renegotiating salaries with only some of them. The advantage of pooling together agents and threats suggests that employers are better positioned to exploit workers as the size of their workforce increases. When market conditions worsen, a big firm would possess more threats than a smaller firm even as both firms experience the same (percentage) decline in demand. As we have shown, although big firms also face a higher number of employees, their ability to credibly threaten them will often be greater. The current protection granted to workers could thus be least effective when it is perhaps most needed.

Countries as well resort to threats to promote their interests. International negotiations are commonly held in the shadow of explicit threats of military strikes should consensus not reached. A common feature of countries’ behavior in this context is reliance on military alliances. Under such agreements, the contracting countries commit to assist any member country involved in an armed conflict. By joining an alliance, therefore, a country undertakes to make the adversaries of its allies its own.¹⁸

A natural question arises, however, on the strategic wisdom of expanding a country’s circle of adversaries in exchange for reciprocal commitments to join forces. Upon forming an alliance, a country might be embroiled in military clashes with countries with which it has no direct conflict.¹⁹ The rationale for military alliances becomes particularly elusive when member countries are heterogeneous in terms of their adversaries or military force. A country whose adversaries pose mild risks is arguably worse off entering an alliance with a country (of similar might) facing stronger rivals. Yet alliances often exhibit considerable variations among members (See Morrow 1991).

The benefit of uniting threats and opponents suggests that alliances can serve countries at the negotiation table. A belligerent adversary will agree to settle only in the face of a credible threat to resort to forcible measures should negotiations fail. Because countries confronting multiple opponents often hold sequential negotiations, the formation of an alliance can enhance the credibility of military threats. In particular, forming a military alliance amounts to uniting threats and opponents and can therefore toughen a country’s bargaining position in peace negotiations.

¹⁸See, for example article 5 of NATO’s charter: "The Parties agree that an armed attack against one or more of them ... shall be considered an attack against them all and consequently they agree that ... each of them ... will assist the Party or Parties so attacked ... by the use of armed force."

¹⁹See Bloch (2011) and Sandler (1993) for surveys of the economic literature on the incentives to form military alliances. While the current literature has suggested that strategic concerns can motivate the formation of alliances, it has considered signaling (Morrow 1994, Fearon 1997) and commitment (Smith 1995) arguments, which are unrelated to our analysis.
This "credibility advantage" implies that countries can gain from entering a military alliance even if their adversaries and military capabilities are manifestly different. The example in the Introduction, involving one principal (A) facing agents of 16, 15, and 14, and another (B) facing agents of 6, 5 and 4, can help illustrate this point. Suppose the principals are two countries each possessing a single threat (ability to win one armed conflict), and that agents are contiguous opponents characterized by the magnitude of harm that each could inflict. Although country A’s direct opponents are considerably more harmful, both countries gain from forming a military alliance. By joining forces and making country A’s adversaries its own, country B—which faces less harmful adversaries—can now use the alliance’s joint threats of military engagements to overcome both its original and assumed opponents.

Finally, the fact that military treaties strengthen countries’ bargaining position indicates that the incentives to forge alliances may be broader than has been proposed previously. Theories of international collaboration conventionally postulate that alliances’ "primary function is to pool military strength against a common enemy" (Snyder 2007, p. 3). The aim of counteracting an expansionist rival is accordingly considered a common prerequisite for countries’ decision to cooperate (Waltz 1979, Walt 1987, Morrow 1991). Our analysis shows, however, that military collaboration benefits countries even in the absence of a common enemy. It is precisely the presence of dispersed adversaries, rather than a central rival, that makes collaboration attractive. By joining forces and opponents, each country can leverage its constrained military force to mount credible threats against both its direct and indirect opponents.

6 Conclusion

Divide-and-conquer, the strategy of dividing opponents and tackling them one by one, is considered conducive to the exercise of power. This paper shows the advantage of the opposite strategy—uniting opponents and coping with them together—when it comes to the optimal use of threats. In the presence of limited resources and constraints on negotiations, combining opponents never weakens and often boosts the credibility of threats.
Appendix

This Appendix restates and proves Definition 1 and Theorem 3.

**Definition 2** A minimal $\tau^k$ sequence is a sequence whose last $2k$ terms are 1 and each preceding term is $x_i = S(X_{i+1}) = 2x_{i+k}$.

**Proof.** By Corollary 1, any sequence of $2k$ terms is threatenable with $k$ threats. Thus the last $2k$ terms of a minimal $\tau^k$ sequence are each equal to 1. To show that $x_i = 2x_{i+k}$ for each preceding term, we proceed by induction.

Consider $k = 1$. Because $X^{D(0)} = X$, in a minimal $\tau^1$ sequence $x_i = S(X_{i+1})$ and $x_{i+1} = S(X_{i+2})$ for $i \leq n - 2$. Subtracting the second equation from the first gives $x_i - S(X_{i+2}) = S(X_{i+1}) - x_{i+1}$. Now, $S(X_{i+1}) - x_{i+1}$ equals $S(X_{i+2})$ (by definition of $S(\cdot)$). We accordingly obtain $x_i = 2S(X_{i+2})$. But $x_{i+1} = S(X_{i+2})$ in turn implies that $x_i = 2x_{i+1}$. Thus, any non-1 term is equal to twice the subsequent term. The sum of all the terms, $S(X_1)$, is therefore $2x_1$.

For the induction step, suppose that a minimal $\tau^j$ sequence consists of $2j$ terms of 1 preceded by bunches of $j$ consecutive terms of 2, 4, 8, ... More specifically, suppose that the $i$ term of a minimal $\tau^j$ sequence is 1 if $i = n - 2j + 1, ..., n$ and $l$ if $i = n - (l+1)j + 1, ..., n-lj$, where $l = 2, 4, 8, ...$ Then, a minimal $\tau^{j+1}$ sequence consists of $2j + 2$ terms of 1 preceded by bunches of $j + 1$ consecutive terms of 2, 4, 8, ...; that is, the $i$ term of a minimal $\tau^{j+1}$ sequence is 1 if $i = n - 2(j + 1) + 1, ..., n$ and $l$ if $i = n - (l+1)(j + 1) + 1, ..., n-l(j + 1)$, where $l = 2, 4, 8, ...$

Let $X$ be a minimal $\tau^{j+1}$ sequence. Then, by definition of a minimal threatenable sequence, $x_i = X_{i+1}^{D(j)}$ for $i \leq n - 2j - 2$. Because the last $2j + 2$ terms are each equal to 1, it follows that $X_{n-i}^{D(j-1)} = 2$ for $i = n - 3(j + 1) + 1, ..., n - 2(j + 1)$ (that is, for the $j + 1$ terms preceding the last $2j + 2$ terms). To see why, observe that by dropping two terms out of the last $2j + 2$ terms of 1, the remaining subsequence becomes, by the induction hypothesis, a minimal $\tau^j$ sequence and therefore is threatenable with $j$ threats. This implies that the $j + 1$ terms preceding the last $2j + 2$ terms ($x_{n-3(j+1)+1}, ..., x_{n-2(j+1)}$) are each equal to 2. Similarly, for the $j + 1$ preceding terms ($x_{n-4(j+1)+1}, ..., x_{n-3(j+1)}$), we have $X_{n-i}^{D(j-1)} = 4$. This follows because by dropping two terms out of the last $2j + 2$ terms of 1 and an additional term of 2 out of the preceding $j + 1$ terms of 2, the remaining subsequence becomes, by the induction hypothesis, a minimal $\tau^j$ sequence and therefore is threatenable with $j$ threats. In general, to render the subsequence $X_i$ threatenable with $j$ threats, where $i = n - (l+1)(j + 1) + 1, ..., n-l(j + 1)$, where $l = 2, 4, 8, ...$ the prosecutor must give up on $2^{l-1}$ (that is, $X_i^{D(j-1)} = 2^{l-1}$). We thus have $x_i = 2x_{i+(j+1)}$.

**Theorem 3** The LUB of the prosecutor’s percentage loss from dividing a $\tau^k$ sequence $X$ into $k$ subsequences is decreasing with $k$ and is equal to $\frac{k-1}{2k^2}$.

20More specifically, $S(X_1) = x_n + x_{n-1}(2^{n-1} - 1)$, where the second term is the sum of a geometric series with $n - 1$ terms and a common ratio of 2. Plugging 1 for both $x_n$ and $x_{n+1}$ gives $S(X_1) = 2^{n-1}$. But $x_1 = 2^{n-2}$ and therefore $S(X_1) = 2x_1$. 

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Proof. The proof proceeds by generalizing (3), (4), and (5) to an arbitrary $k$ in the following three Lemmas.

**Lemma A1** If $X$ is a $\tau^k$ sequence and $X_{2k}$ is a $\tau^1$ sequence, then $X$ is partitionable.

**Proof.** By induction. From (3), the Lemma is true for $k = 2$. We proceed by showing that if the Lemma is true for $j \geq 2$, then it is true for $j + 1$ as well.

Let $X$ be a $\tau^{j+1}$ sequence and suppose that $X_{2j+2}$ is threatenable with one threat. We will show that $X_2^{T(j)}$ is partitionable into $j$ subsequences and that $x_1 \cup X_2^{D(j)}$ is threatenable with one threat. It will then follow that $X$ is partitionable into $j + 1$ subsequences.

Let $X_{2j+1} = (x_i, \ldots, x_{2j+1})$ be a subsequence consisting of the $2j$ consecutive terms of $X$ beginning with $x_2$. By Corollary 1, $X_{2j+1}$ is threatenable with $j$ threats. We now consider two cases.

Case (i): $X_{2j+1} \subseteq X_2^{T(j)}$. In this case, $(X_2^{T(j)})_{2j} \subseteq X_{2j+2}$, where $(X_2^{T(j)})_{2j}$ is a subsequence of all the terms of $X_2^{T(j)}$ starting with the $(2j)$th term. Because $X_{2j+2}$ is threatenable with one threat, so is $(X_2^{T(j-1)})_{2j+2}$. It follows, by the induction hypothesis, that $X_2^{T(j)}$ is partitionable into $j$ subsequences. In addition, $X_{2j+1} \subseteq X_2^{T(j)}$ implies that $X_2^{D(j)} \subseteq X_{2j+2}$. Because $X_{2j+2}$ is threatenable with one threat, so is $x_1 \cup X_2^{D(j)}$.

Case (ii) $X_{2j+1} \nsubseteq X_2^{T(j)}$. In this case, $X_{2j} \subseteq X_2^{T(j-1)}$. This follows because $X_{2j+2}$ is a $\tau^1$ sequence and therefore $x_{2j+2} \geq S(X_{2j+3})$. Adding $x_{2j+2}$ to both sides gives $2x_{2j+2} \geq S(X_{2j+2})$, which implies that $x_{2j} + x_{2j+1} \geq S(X_{2j+2})$. If $(x_{2j}, x_{2j+1}) \subseteq X_2^{D(j)}$, the prosecutor could swap all the terms of the intersection $X_2^{T(j)} \cap X_{2j+2}$ for $x_{2j}$ and $x_{2j+1}$. Here too, therefore, $(X_2^{T(j)})_{2j} \subseteq X_{2j+2}$; hence, by the induction hypothesis, $X_2^{T(j)}$ is partitionable into $j$ subsequences. In addition, the first term of $X_2^{D(j)}$ is $x_{2j+1}$, where every other term belongs to $X_{2j+2}$ (which is a $\tau^1$ sequence). Because $X_2^{D(j)}$ is threatenable with one threat, so is $x_1 \cup X_2^{D(j)}$. $\blacksquare$

It follows from Lemma A2 that upon dividing a $\tau^k$ into $k$ subsequences, the prosecutor never loses more than the minimum sum of terms that must be dropped to render $X_{2k}$ threatenable with one threat. This is because by dropping the terms of $X_4^{D(1)}$, the subsequence $X_4$ of the resulting sequence becomes a $\tau^1$ sequence. The next Lemma states that formally:

**Lemma A2** The maximum absolute loss from dividing a $\tau^k$ sequence $X$ into $k$ subsequences is $S(X_{2k}^{D(1)})$.

We now consider the maximum percentage loss from rendering a $\tau^k$ sequence threatenable with $k - 1$ threats:

**Lemma A3** Let $X$ be a sequence threatenable with $k$ threats. Then the prosecutor must never drop more than $1/k \times S(X)$ to render $X$ threatenable with $k - 1$ threats.
Proof. We begin by finding an upper bound on \( S(X^{D(k-1)}) \)—the minimum sum of terms that must be dropped to render \( X \) threatenable with \( k-1 \) threats—in terms of \( x_1 \), the first term of \( X \). Because \( X \) is threatenable with \( k \) threats, it must satisfy Condition \( \tau^k \). This implies, in particular, that

\[
x_1 \geq S(X_2^{D(k-1)}). \quad (*)
\]

Moreover, \( S(X_1^{T(k-1)}) \geq S(X_2^{T(k-1)}) \), because the prosecutor can do just as well with \( X_1 \) as with \( X_2 \) by dropping \( x_1 \). We thus have \( S(X) - (X^{D(k-1)}) \geq S(X_2) - S(X_2^{D(k-1)}) \), where the left-hand side is \( S(X_1^{T(k-1)}) \) and the right-hand side is \( S(X_2^{T(k-1)}) \). Substituting \( x_1 \) for the difference \( S(X_1) - S(X_2) \) gives \( S(X_2^{D(k-1)}) + x_1 \geq S(X^{D(k-1)}) \). Setting \( S(X_2^{D(k-1)}) \) equal to its maximum value of \( x_1 \) [from (\*)], we get

\[
2x_1 \geq S(X^{D(k-1)}). \quad (**)
\]

Thus, to render a \( \tau^k \) sequence threatenable with \( k-1 \) threats the prosecutor must never drop more than \( 2x_1 \).

Next, we set \( S(X^{D(k-1)}) \) equal to its maximum value of \( 2x_1 \) [from (**)]. We proceed by constructing a \( \tau^{k-1} \) sequence \( X^{T(k-1)} \) such that \( X^{T(k-1)} \cup X^{D(k-1)} \) is threatenable with \( k \) threats and \( S(X^{T(k-1)}) \leq S(X') \) for any \( \tau^{k-1} \) sequence \( X' \) such that \( X' \cup X_1^{D(k-1)} \) is threatenable with \( k \) threats. Thus, \( X^{T(k-1)} \) is the lowest-sum-of-terms \( \tau^{k-1} \) sequence whose union with \( X^{D(k-1)} \) is threatenable with \( k \) threats.

Because \( X^{T(k-1)} \) is threatenable with \( k-1 \) threats, by Corollary 2 \( x_1 + \ldots + x_{k-1} \geq S(X_{k-1}^{T(k-1)}) \). Adding \( S(X_k^{T(k-1)}) \) to each side gives \( S(X^{T(k-1)}) \geq 2S(X_k^{T(k-1)}) \). Thus \( S(X^{T(k-1)}) \) is bounded below by \( 2S(X_k^{T(k-1)}) \). But for the union \( X_1^{T(k-1)} \cup X_1^{D(k-1)} \) to be threatenable with \( k \) threats, we must also have (again, by Corollary 2) \( x_1 + \ldots + x_k \geq S(X_{k+1}^{T(k-1)}) + 2x_1 \), where \( 2x_1 = S(X^{D(k-1)}) \). Adding \( S(X_k^{T(k-1)}) \) to both sides gives \( S(X^{T(k-1)}) \geq 2S(X_k^{T(k-1)}) + 2x_1 \). Substituting \( S(X^{T(k-1)}) - x_k \) for \( S(X_{k+1}^{T(k-1)}) \) yields \( S(X^{T(k-1)}) \geq 2S(X_k^{T(k-1)}) + 2x_1 - 2x_k \). Now, \( 2x_1 \geq 2x_k \) and therefore \( S(X^{T(k-1)}) = \not{2S(X_k^{T(k-1)})} \) iff \( x_1 = x_k \). But if \( x_1 = x_k \) then \( S(X_k^{T(k-1)}) = (k-1)x_1 \), and \( S(X) = 2kx_k \). The ratio \( \frac{S(X_1^{D(k-1)})}{S(x)} \) is therefore bounded above by \( \frac{2x_k}{2kx_k} = \frac{1}{k} \).

Repeated application of Lemma A3 yields the following Lemma:

**Lemma A4** Let \( X \) be a \( \tau^k \) sequence. Then the prosecutor must never drop more than \( (k-j)/k \times S(X) \) to render \( X \) threatenable with \( j \) threats.

Observe that Lemma A4 is satisfied as equality iff \( X \) is a minimal \( \tau^k \) sequence and \( n = ak \), where \( a \) is an integer greater than 2. For example, in rendering the 9-term minimal \( \tau^3 \) sequence \( X = (2, 2, 2, 1, 1, 1, 1, 1, 1) \) threatenable with two threats, the prosecutor’s percentage loss is 2/3 (because \( S(X^{T(2)}) = 8 \) and \( X^{D(2)} = 4 \)). Similarly, in rendering the 12-term minimal \( \tau^4 \) sequence \( X = (2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1) \) threatenable with three threats, the prosecutor’s percentage loss is 1/4 (because \( S(X^{T(3)}) = 12 \) and \( X^{D(3)} = 4 \)).

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We conclude the proof by constructing an unpartitionable \( \tau^k \) sequence \( U \) such that 
\[
S(U^{D(1)}) = \frac{k-1}{k}S(U^{T(1)})
\] (by Lemma A4) and all the terms of \( U_{2k}^{D(1)} \) must be dropped upon dividing \( U \) into \( k \) subsequences (otherwise the percentage loss from division would not be maximal). For such a sequence \( U \) we must have
\[
S(U) > 2kS(U_{2k}) - (2k - 2)u_n.
\]
where \( u_n \) is the last—and therefore smallest—term of \( U_{2k}^{D(1)} \). To see this, observe first that the prosecutor must give up on all the terms of \( U_{2k}^{D(1)} \) iff \( u_1 - u_{2k-1} < u_n \). For if \( u_1 - u_{2k-1} \geq u_n \), then \( U \) could be partitioned into \( k \) subsequences \( (u_1, u_{2k-1}, u_n), (u_2, u_3), (u_4, u_5), ..., u_{2k-2} \cup U_{2k}^{T(1)} \), each attackable with one attack. The subsequence \( u_{2k-2} \cup U_{2k}^{T(1)} \) is a \( \tau^{k-1} \) sequence because \( U_{2k}^{T(1)} \) is attackable with one attack and therefore \( u_{2k-1} \geq S(U_{2k-1}^{T(1)}) - u_{2k-1} \). In addition, \( S(U_{2k-1}^{T(1)}) \geq S(U_{2k}^{T(1)}) \). But \( u_{2k-2} \geq u_{2k-1} \) and therefore \( u_{2k-2} \geq S(U_{2k}^{T(1)}) \).

As the number of terms of \( U_{2k} \) goes to infinity, each of the \( 2k - 1 \) terms of the head subsequence approaches \( S(U_{2k}) \) and the sum of all the terms of \( U \) approaches \( 2kS(U_{2k}) \). The percentage loss from division is accordingly bounded above by \( \frac{(k-1)/k}{2k} \).
References


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